

Scalar Casimir effect between two concentric D -dimensional spheres

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Abstract

The Casimir energy for a massless scalar field between the closely spaced two concentric D -dimensional (for $D > 3$) spheres is calculated by using the mode summation with contour integration in the complex plane of eigenfrequencies and the generalized Abel-Plana formula for evenly spaced eigenfrequency at large argument. The sign of the Casimir energy between closely spaced two concentric D -dimensional spheres for a massless scalar field satisfying the Dirichlet boundary conditions is strictly negative. The Casimir energy between $D - 1$ dimensional surfaces close to each other is regarded as interesting both by itself and as the key to describing of stability of the attractive Casimir force.

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1 INTRODUCTION

The measurable consequences of the macroscopic phenomena in the quantum theory of fields is Casimir effect. This effect due to the vacuum polarization of the quantized field was originally derived by H. B. G. Casimir [1]. He calculated the negative renormalized quantum vacuum energy for the electromagnetic field bounded by two uncharged parallel plates. He concluded that there must exist an attractive force between the plates. The attractive force: $F = -\frac{\pi^2 \hbar c}{240a^4}$ which is expressed in terms of the Planck constant \hbar , the velocity of light c , and the distance between the plates a , should act on unit area of two uncharged conducting plane parallel plates in vacuum. This attractive force was confirmed experimentally [2, 3, 4]. Both theoretical and experimental studies of the Casimir effect can provide great insight into understanding the nature of the quantum vacuum [5, 6, 7, 8, 9]

At first time, having found the negative energy for uncharged conducting parallel plates due to Casimir effect, the hope was what the same would appear

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for the spherical geometry. This expectation was shattered by Boyer [10]. Boyer first showed Casimir energy for the spherical shell is positive. The sign of the positive Casimir energy produces the repulsive force. Boyer's result has been later confirmed by using different regularization techniques [11, 12, 13]. Nowadays, the nature of the Casimir energy is the strong dependence on the geometry of the spacetime, the dimension of the spacetime and on the boundary condition imposed [7, 14, 15]. Recently, the Casimir energy between two concentric spheres and cylinders in $D = 3$ dimension have been considered by using the different regularization methods [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31] for the scalar and electromagnetic fields. Moreover, the investigation of the dimensional dependence of the Casimir energy is of interest. The Casimir energy in the higher dimensional spacetime has a long history. One of the pioneering works is that the Casimir energy for a massless scalar field and a massless vector field in a D -dimensional rectangular cavity were derived by Ambjorn and Wolfram [32]. Afterwards, the scalar and electromagnetic Casimir energy for a D -dimensional sphere were calculated using Green's functions method and zeta function regularization [33, 34, 35, 36, 37], and the Casimir interaction between two concentric spheres in D -dimensions for the scalar fields with Robin boundary condition and the electromagnetic fields at finite temperature have been studied [38, 39]. In particular today, physical theories and models with extra dimensions are active areas of research, for example String theory and Brane world models.

As matter of fact, the sign and magnitude of the Casimir energy may strongly depend on (a) the spacetime dimensionality, (b) the type of the boundary conditions, (c) type of the fields, (d) the lengths between the surfaces is critically dependent on their nanometre-scale shape, (e) the curved spacetime background, (f) compactness spacetime, and (g) the finite temperature [7, 40]. In this work, the consequences of (c), (e) and (g) will not be considered, and we consider the Casimir energy between closely spaced two concentric D -dimensional spheres for a massless scalar field satisfying the Dirichlet boundary conditions. From a mathematical point of view, a simple method for calculating the Casimir energy between closely spaced two concentric D -dimensional spheres for a massless scalar field is developed which is based on a direct mode summation with the contour integration in a complex plane of eigenfrequencies and using the generalized Abel-Plana sum formula for evenly spaced eigenfrequency at large argument. One of the motivations of our work for us to perform the Casimir energy calculation for a massless scalar field between $D - 1$ dimensional surfaces close to each other is that the physics in higher dimensional spacetime have become a trend since the existence of the extra dimensions allows the solving of the some fundamental problems in physics as the hierarchy problem.

The organization of this paper is as follows. In section 2, The Casimir energy of a massless scalar field subjected to the Dirichlet boundary conditions on between closely spaced two concentric D -dimensional spheres is calculated without any approximation techniques. Concluding remarks and discussion of the Casimir energy for a massless scalar field in an annular region of D -dimensional geometry is presented in section 3.

The units are such that $\hbar = c = 1$.

2 CASIMIR ENERGY

We begin with a massless scalar field considered in a D -dimensional spherical geometry, where the metric is given by

$$\begin{aligned} ds^2 &= dt^2 - d\sigma^2, \quad \text{where} \\ d\sigma^2 &= dx_1^2 + dx_2^2 + dx_3^2 + \dots + dx_D^2. \end{aligned} \quad (1)$$

In spherical coordinates

$$\begin{aligned} x_1 &= r \sin \theta_1 \sin \theta_2 \sin \theta_3 \dots \sin \theta_{D-2} \cos \phi \\ x_2 &= r \sin \theta_1 \sin \theta_2 \sin \theta_3 \dots \sin \theta_{D-2} \sin \phi \\ &\vdots \\ x_{D-1} &= r \sin \theta_1 \cos \theta_2 \\ x_D &= r \cos \theta_1. \end{aligned} \quad (2)$$

Where $0 \leq \theta_j \leq \pi$, $j = 1, 2, 3, \dots, D-2$ and $0 \leq \phi \leq 2\pi$. $x_1^2 + x_2^2 + x_3^2 + \dots + x_D^2 = r^2$ is defined by D dimensional hypersphere. The metric becomes

$$ds^2 = dt^2 - [dr^2 + r^2 d\theta_1^2 + r^2 \sin^2 \theta_1 d\theta_2^2 + \dots + r^2 \sin^2 \theta_1 \sin^2 \theta_2 \dots \sin^2 \theta_{D-2} d\phi^2]. \quad (3)$$

Massless scalar field satisfies the Klein Gordon equation in this geometry is given by

$$\square \Psi(t, r, \vartheta, \phi) = 0 \quad (\text{where } \vartheta = \theta_1, \theta_2, \dots, \theta_{D-2}). \quad (4)$$

Here \square is the D'Alembertian operator associated with the metric given by the line element in Eq. (3). Solution of Eq. (4) could be easily found by using the method of separation of variables and is written as (for $D \geq 4$)

$$\begin{aligned} \Psi(t, r, \vartheta, \phi) &= \sum_{\{\lambda\}} e^{-i\omega t} \left[A_k r^{-\frac{(D-2)}{2}} J_\nu(\omega r) + B_k r^{-\frac{(D-2)}{2}} N_\nu(\omega r) \right] \\ &\quad \left\{ \prod_{\mu=1}^{D-3} \sin^{M_{D-\mu-2}} \theta_\mu C_{M_{D-\mu-1}-M_{D-\mu-2}}^{M_{D-\mu-2}+\frac{(D-\mu-1)}{2}}(\cos \theta_\mu) \right\} Y_{\ell m}(\theta_{D-2}, \phi). \end{aligned} \quad (5)$$

Where $\nu = k + \frac{(D-2)}{2}$ and $\{\lambda\}$ refers to $m, \ell (= M_1), M_2, M_3, M_4, \dots, M_{D-3}$, and $k (= M_{D-2})$. D is the number of dimension of sphere. $J_\nu(\omega r)$ and $N_\nu(\omega r)$ are Bessel functions first and second kind, respectively. And the function $C_{M_{D-\mu-1}-M_{D-\mu-2}}^{M_{D-\mu-2}+\frac{(D-\mu-1)}{2}}(\cos \theta_\mu)$ corresponds to the Gegenbauer or ultraspherical polynomials. $Y_{\ell m}(\theta_{D-2}, \phi)$ are the spherical harmonics. We note that the spherical boundary condition at $\Psi(t, r=a, \vartheta, \phi) = 0$ and $\Psi(t, r=b, \vartheta, \phi) = 0$ (a (b) is inner (outer) radius of D -dimensional sphere) has not imposed on Eq. (5) yet. Hence, ω still remains a continuous parameter, while $m, \ell (= M_1), M_2, M_3, \dots, k (= M_{D-2})$ take the values

$$\begin{aligned}
m &= -\ell, -\ell+1, \dots, 0, 1, 2, \dots, \ell-1, \ell, \\
\ell (= M_1) &= 0, 1, 2, 3, 4, \dots, M_2 \\
M_2 &= 0, 1, 2, 3, 4, \dots, M_3 \\
M_3 &= 0, 1, 2, 3, 4, \dots, M_4 \\
&\vdots \\
&\vdots \\
&\vdots \\
M_{D-4} &= 0, 1, 2, 3, 4, \dots, M_{D-3} \\
M_{D-3} &= 0, 1, 2, 3, 4, \dots, M_{D-2} \\
k (= M_{D-2}) &= 0, 1, 2, 3, 4, \dots
\end{aligned} \tag{6}$$

We now impose the boundary conditions for D -dimensional spherical geometry i.e.,

$$\Psi_{\omega \ell m}(t, r=a, \vartheta, \phi) = 0, \text{ and similarly } \Psi_{\omega \ell m}(t, r=b, \vartheta, \phi) = 0. \tag{7}$$

The eigenfunction that satisfy the boundary conditions is

$$\begin{aligned}
\Psi_{\omega \ell m}(t, r, \vartheta, \phi) &= c_0 \sum_{\{\lambda\}} e^{-i\omega t} r^{-\frac{(D-2)}{2}} \left[J_\nu(\omega r) - \frac{J_\nu(\omega a)}{N_\nu(\omega a)} N_\nu(\omega r) \right] \\
&\quad \left\{ \prod_{\mu=1}^{D-3} \sin^{M_{D-\mu-2}} \theta_\mu C_{M_{D-\mu-1}-M_{D-\mu-2}}^{M_{D-\mu-2}+\frac{(D-\mu-1)}{2}}(\cos \theta_\mu) \right\} Y_{\ell m}(\theta_{D-2}, \phi).
\end{aligned} \tag{8}$$

Where c_0 is the normalization constant and ω is the root of the following transcendental equation

$$J_\nu(\omega b) N_\nu(\omega a) - J_\nu(\omega a) N_\nu(\omega b) = 0 \text{ where } \nu = k + \frac{(D-2)}{2}. \tag{9}$$

We define the Casimir energy between two concentric D -dimensional spheres for a massless scalar field

$$\begin{aligned}
E_C &= \frac{1}{2} \sum_{\{\lambda\}} \omega_\lambda, \\
&= \frac{1}{2} \sum_{k=0}^{\infty} \left\{ \prod_{\mu=1}^{D-3} \sum_{M_\mu=0}^{M_{\mu+1}} \right\} \sum_{m=-\ell}^{\ell} \sum_{n=1}^{\infty} \omega_{nk}, \\
&= \frac{1}{2} \sum_{k=0}^{\infty} g^{(D)}(\nu) \sum_{n=1}^{\infty} \omega_{nk}
\end{aligned} \tag{10}$$

Where $\omega_{n\ell}$ are eigenfrequencies stems from root of the transcendental equation given in Eq. (9) and the degeneracy of each eigenfrequency $g^{(D)}(\nu)$ (D represents space dimension) could be written as

$$g^{(D)}(\nu) = 2\nu \frac{[\nu + \frac{1}{2}(D-4)]!}{(D-2)! [\nu - \frac{1}{2}(D-2)]!} \quad \text{and} \quad \nu = k + \frac{(D-2)}{2}. \tag{11}$$

The zeros of the frequency equation are real and simple since ν is real and a and b positive. We know that Bessel's series equation are convergent for all values of the argument. When $|\omega|$ is very large the convergence is so slow. To render the series useless for the frequency calculation we need the rapidly convergent evaluation of the Bessel's function formula. A very rapidly convergent evaluation of the frequency equation can be obtained by using the uniform asymptotic expansions for the Bessel function. Hence, we should examine the behavior of the eigenfrequency spectrum for large arguments at fixed k and large order as $k \rightarrow \infty$. Thus, to carry out the summation with respect to k in E_C , the sum $\sum_{n=1}^{\infty} \omega_{nk}$ given in equation (10) replaced by $\sum_{n=1}^{\infty} \bar{\omega}_{nk} + \sum_{n=1}^{\infty} \tilde{\omega}_{nk}$ where $\tilde{\omega}_{nk}$ is the eigenvalue spectrum of the limit $\omega \rightarrow \infty$ at fixed k [10, 18, 19]. Then, the Casimir energy which is defined by the eigenfrequency spectrum for large arguments at fixed k and large order as $k \rightarrow \infty$ and can be written as

$$E_C = \frac{1}{2} \sum_{k=0}^{\infty} g^{(D)}(\nu) \left(\sum_{n=1}^{\infty} \bar{\omega}_{nk} + \sum_{n=1}^{\infty} \tilde{\omega}_{nk} \right). \tag{12}$$

Now, we calculate the eigenfrequencies for large arguments at fixed ν . We employed the Hankel's asymptotic expansion [18, 19, 41] when ν is fixed, $\omega a \gg 1$ and $\omega b \gg 1$, one obtains

$$J_\nu(\tilde{\omega}a) \simeq \sqrt{\frac{2}{\pi \tilde{\omega}a}} \left[\cos\left(\tilde{\omega}a - \frac{\nu}{2}\pi - \frac{\pi}{4}\right) - \frac{(4\nu^2 - 1)}{8\tilde{\omega}a} \sin\left(\tilde{\omega}a - \frac{\nu}{2}\pi - \frac{\pi}{4}\right) \right] \tag{13}$$

$$N_\nu(\tilde{\omega}a) \simeq \sqrt{\frac{2}{\pi\tilde{\omega}a}} \left[\cos\left(\tilde{\omega}a - \frac{\nu}{2}\pi - \frac{\pi}{4}\right) + \frac{(4\nu^2 - 1)}{8\tilde{\omega}a} \sin\left(\tilde{\omega}a - \frac{\nu}{2}\pi - \frac{\pi}{4}\right) \right] \quad (14)$$

and similar expressions for $J_\nu(\tilde{\omega}b)$ and $N_\nu(\tilde{\omega}b)$ with a interchanged for b . Putting (13) and (14) in the frequency equation given by (9), we obtain the zeros of frequency equation are almost evenly spaced for very large argument at fixed ν .

$$\tilde{\omega}_{nk}^2 \simeq \left(\frac{n\pi}{b-a} \right)^2 + \frac{\nu^2}{ab} \quad \text{where } n = 1, 2, 3, 4, 5, 6, \dots \quad (15)$$

And, the frequency equation $\bar{\omega}_{nk}$ the first sum is given in Eq. (12) as the uniform asymptotic expansion of the Bessel function can be written as

$$f_\nu(\nu\bar{\omega}, \lambda) = J_\nu(\nu\bar{\omega}) N_\nu(\nu\bar{\omega}\lambda) - J_\nu(\nu\bar{\omega}\lambda) N_\nu(\nu\bar{\omega}) \quad (16)$$

Where $\lambda = \frac{a}{b}$ ($b > a$) and $\nu = k + \frac{(D-2)}{2}$.

Then, the Casimir energy between the closely spaced two concentric D -dimensional spheres for a massless scalar field which is defined by the eigenfrequency spectrum for large arguments at fixed k and large order as $k \rightarrow \infty$ can be written as [18, 19]

$$\begin{aligned} E_C^{(D)} &= \frac{1}{2} \sum_{k=0}^{\infty} g^{(D)}(\nu) \sum_{n=1}^{\infty} \bar{\omega}_{nk} + \frac{1}{2} \sum_{k=0}^{\infty} g^{(D)}(\nu) \sum_{n=1}^{\infty} \sqrt{\left(\frac{n\pi}{b-a} \right)^2 + \frac{\nu^2}{ab}} \\ &= \bar{E}_C^{(D)} + \tilde{E}_C^{(D)}. \end{aligned} \quad (17)$$

Where $\bar{\omega}_{nk}$ is the root of the frequency equation given in Eq. (16).

We consider the first sum defined in Eq. (17). This divergent expression can be rendered finite by the use of a cutoff or convergence factor. Then we define the first sum,

$$\begin{aligned} \bar{E}_C^{(D)} &= \frac{1}{2} \sum_{k=0}^{\infty} g^{(D)}(\nu) \sum_{n=1}^{\infty} \omega_{nk} e^{-\alpha\omega_{nk}} \\ &= \frac{1}{2} \sum_{k=0}^{\infty} g^{(D)}(\nu) S_k, \end{aligned} \quad (18)$$

where ω_{nk} refers to $\bar{\omega}_{nk}$ and the factor of $e^{-\alpha\omega_{nk}}$ plays the role of an exponential regulator which effectively suppresses the high frequency contributions to the Casimir energy, and $S_k = \sum_{n=1}^{\infty} \omega_{nk} e^{-\alpha\omega_{nk}}$ is generated by the frequency equation (16). To evaluate the sum S_k , we use the integral representation from the Cauchy's theorem [13, 18, 19, 26] that for two functions $f_k(z)$ and

$\phi(z)$ analytic within a closed contour C in which $f_k(z)$ has isolated zeros at $x_1, x_2, x_3, \dots, x_n$,

$$\frac{1}{2\pi i} \oint_C dz \phi(z) \frac{d}{dz} \ln f_k(z) = \sum_j \phi(x_j) . \quad (19)$$

We choose $\phi(z) = z e^{-\alpha z}$ where α is a real positive constant thus leads to

$$\frac{1}{2\pi i} \oint_C dz e^{-\alpha z} z \frac{d}{dz} \ln f_\ell(z) = \sum_j z_j e^{-\alpha z_j} . \quad (20)$$

Using this result to replace the sum S_k by a contour integral, the first term of the Casimir energy becomes

$$\overline{E}_C = \frac{1}{2} \sum_{k=0}^{\infty} g^{(D)}(\nu) \frac{1}{2\pi i} \oint_C dz e^{-\alpha z} z \frac{d}{dz} \ln f_\nu(\nu z, \lambda) , \quad (21)$$

where the frequency function $f_\nu(\nu z, \lambda)$ is given Eq. (16). The contour C encloses all the positive roots of the equation $f_\nu(\nu z, \lambda) = 0$. This contour can be conveniently broken into three parts [13, 18, 19, 26]. These consist of a circular segment C_Γ and two straight line segments Γ_1 and Γ_2 forming an angle ϕ and $\pi - \phi$ with respect to the imaginary axis. When the radius Γ is fixed, the contour C_Γ encloses a finite number of roots of the equation $f_\nu(\nu z, \lambda) = 0$. Since the sum of these roots is obviously infinite, the radius Γ is a regularization parameter, and taking the limit $\Gamma \rightarrow \infty$ (when $\alpha > 0$) means the removal of the regularization, the contribution of C_Γ vanishes provided that $\phi \neq 0$. Hence the exponential regulator in the Cauchy integral plays the role of the eliminate of the contribution to the circular part of the contour integral. Taking the contributions along Γ_1 and Γ_2 which are complex conjugates of each other and rescaling of integration variable, then Eq. (18) becomes [12, 13]

$$\overline{E}_C^{(D)} = -\frac{1}{2\pi b} \lim_{\alpha \rightarrow 0} \sum_{k=0}^{\infty} g^{(D)}(\nu) \operatorname{Re} e^{-i\phi} \int_0^\infty dy e^{-i\alpha y e^{-i\phi}/b} y \frac{d}{dy} \ln f_\nu(\nu y e^{-i\phi}, \lambda) . \quad (22)$$

Now we use the Lommel's expansions or the multiplication theorem for the function of $f_\nu(y e^{-i\phi}, \lambda)$ and uniform asymptotic expansion of the modified Bessel functions one obtains [18, 19, 42]

$$\begin{aligned}
y \frac{d}{dy} \ln f_\nu(\nu y e^{-i\phi}, \lambda) &= \frac{(1-\lambda^2)^2}{12} (\nu y e^{-i\phi})^2 + \frac{(1-\lambda^2)^3}{24} (\nu y e^{-i\phi})^2 \\
&+ \frac{(1-\lambda^2)^4}{720} \left[(\nu y e^{-i\phi})^2 (-\nu^2 + 19) - (\nu y e^{-i\phi})^4 \right] \\
&+ \frac{(1-\lambda^2)^5}{1440} \left[-3 (\nu y e^{-i\phi})^2 (\nu^2 - 9) - 2 (\nu y e^{-i\phi})^4 \right] \\
&+ \frac{(1-\lambda^2)^6}{120960} \left[(\nu y e^{-i\phi})^2 (4\nu^4 - 290\nu^2 + 1726) + (\nu y e^{-i\phi})^4 (8\nu^2 - 149) \right. \\
&\left. + 4 (\nu y e^{-i\phi})^6 \right] + [\text{Terms in even powers of } (\nu y e^{-i\phi})], \tag{23}
\end{aligned}$$

where $|\lambda^2 - 1| < 1$ and $\nu = k + \frac{(D-2)}{2}$. Inserting Eq. (23) into Eq. (22) and using the following integral result

$$\begin{aligned}
I(2n) &= e^{-i\phi} \int_0^\infty dy e^{-i\alpha y e^{-i\phi}/b} (y e^{-i\phi})^{2n} \\
&= i (-)^{n+1} (2n)! \left(\frac{b}{\alpha} \right)^{2n+1}, \quad \text{where } n = 0, 1, 2, 3, \dots \tag{24}
\end{aligned}$$

then Eq. (22) becomes

$$\begin{aligned}
\overline{E}_C^{(D)} &= -\frac{1}{2\pi b} \lim_{\alpha \rightarrow 0} \sum_{k=0}^\infty g^{(D)}(\nu) \operatorname{Re} \left\{ i \frac{(1-\lambda^2)^2}{6} \nu^2 \left(\frac{b}{\alpha} \right)^3 + i \frac{(1-\lambda^2)^3}{12} \nu^2 \left(\frac{b}{\alpha} \right)^3 \right. \\
&+ i \frac{(1-\lambda^2)^4}{720} \left[2\nu^2(-\nu^2 + 19) \left(\frac{b}{\alpha} \right)^3 + 24\nu^4 \left(\frac{b}{\alpha} \right)^5 \right] \\
&+ i \frac{(1-\lambda^2)^5}{720} \left[-6\nu^2(\nu^2 - 9) \left(\frac{b}{\alpha} \right)^3 + 48\nu^4 \left(\frac{b}{\alpha} \right)^5 \right] \\
&+ i \frac{(1-\lambda^2)^6}{120960} \left[\nu^2(8\nu^4 - 580\nu^2 + 3452) \left(\frac{b}{\alpha} \right)^3 - 24\nu^4(8\nu^2 - 149) \left(\frac{b}{\alpha} \right)^5 + 2880\nu^6 \left(\frac{b}{\alpha} \right)^7 \right] \\
&\left. + [\text{Terms in imaginary number and even powers of } \nu] \right\}. \tag{25}
\end{aligned}$$

All terms in the above equation have the singular term in the regulator parameter α and purely imaginary. Taking the real part of the parenthesis, thus it leaves the zero result.

$$\overline{E}_C^{(D)} = 0 . \quad (26)$$

The meaning of this result is that there is no contribution from a large order as $k \rightarrow \infty$ modes for the Casimir energy between the closely spaced two concentric D -dimensional spheres.

Thus, D space dimension of the Casimir energy given in Eq. (17) included high eigenfrequency modes i.e. $\omega \rightarrow \infty$ at fixed k can be written as

$$E_C^{(D)} = \frac{1}{2} \sum_{k=0}^{\infty} g^{(D)}(\nu) \sum_{n=1}^{\infty} \sqrt{\left(\frac{n\pi}{b-a}\right)^2 + \frac{\nu^2}{ab}} \quad (27)$$

Where $\nu = k + \frac{(D-2)}{2}$. This divergent sum can be regularized by using the Abel-Plana sum formula which could be given as [6, 8]

$$\begin{aligned} \text{Reg} \left[\sum_{n=1}^{\infty} f(n) \right] &= \\ \sum_{n=0}^{\infty} f(n) - \int_0^{\infty} f(x) dx &= \frac{1}{2} f(0) + i \int_0^{\infty} \frac{f(it) - f(-it)}{e^{2\pi t} - 1} dt. \end{aligned} \quad (28)$$

Where $f(z)$ is an analytic function in the right half plane and Reg refers to the regularized value of the sum. The other useful Abel-Plana sum formula for the half integer number is

$$\sum_{n=0}^{\infty} f\left(n + \frac{1}{2}\right) = \int_0^{\infty} f(x) dx - i \int_0^{\infty} \frac{f(it) - f(-it)}{e^{2\pi t} + 1} dt. \quad (29)$$

We can rewrite the sum given in Eq. (27) using the Abel-Plana sum formula, which leads to

$$\begin{aligned} E_C^{(D)} &= \frac{1}{2} \sum_{k=0}^{\infty} g^{(D)}(\nu) \text{Reg} \left[\sum_{n=1}^{\infty} \sqrt{\left(\frac{n\pi}{b-a}\right)^2 + \frac{\nu^2}{ab}} \right] \\ &= -\frac{1}{4\sqrt{ab}} \sum_{k=0}^{\infty} \nu g^{(D)}(\nu) - \frac{1}{2\pi\sqrt{ab}} \xi \sum_{k=0}^{\infty} \nu^2 g^{(D)}(\nu) \int_1^{\infty} [y^2 - 1]^{\frac{1}{2}} \frac{dy}{e^{\xi\nu y} - 1} . \end{aligned} \quad (30)$$

where $\xi = \frac{2d}{\sqrt{ab}}$ and $d = b - a$. The first divergent sum in Eq. (30) can be removed by using the Hurwitz zeta function [43]

$$\begin{aligned}
\sum_{k=0}^{\infty} \left(k + \frac{(D-2)}{2} \right)^p &= \zeta \left(-p, \frac{(D-2)}{2} \right), \\
&= -\frac{B_{p+1} \left(\frac{(D-2)}{2} \right)}{p+1}.
\end{aligned} \tag{31}$$

Where $p = 0, 1, 2, 3, \dots$ and $B_{p+1} \left(\frac{(D-2)}{2} \right)$ is the Bernoulli polynomial. To regularized the second term in Eq. (30) we use the half integer Abel-Plana sum formula given in Eq. (29) for D -odd space dimension and the Abel-Plana sum formula given in Eq. (28) for D -even space dimension, thus the Casimir energy per unit surface area on the inner sphere for the specialized cases where our space has dimensions $D = 4, 5, 6, 7, 8, 9, 10, 11$ (for $D = 3$ result is given by in [19]) can be written as $\left(\text{the total spherical surface area of } D \text{ dimension is } A^{(D)} = 2\pi^{D/2} \frac{1}{\Gamma(\frac{D}{2})} a^{D-1} \right)$

$$\begin{aligned}
\frac{E_C^{(4)}}{A^{(4)}} &= -\frac{3}{128\pi^2} \left(\frac{\sqrt{ab}}{a} \right)^3 \frac{\zeta(5)}{d^4}, \\
\frac{E_C^{(5)}}{A^{(5)}} &= -\frac{1}{32\pi^3} \left(\frac{\sqrt{ab}}{a} \right)^4 \frac{\zeta(6)}{d^5} \left[1 - \frac{1}{8}\eta^2 \frac{\zeta(4)}{\zeta(6)} \right. \\
&\quad \left. - \frac{7}{64}\eta^4 \frac{\zeta(4)}{\zeta(6)} - \frac{1}{96}\eta^4 \frac{\zeta(2)}{\zeta(6)} \right], \\
\frac{E_C^{(6)}}{A^{(6)}} &= -\frac{15}{1024\pi^3} \left(\frac{\sqrt{ab}}{a} \right)^5 \frac{\zeta(7)}{d^6} \left[1 - \frac{4}{15}\eta^2 \frac{\zeta(5)}{\zeta(7)} \right],
\end{aligned}$$

$$\begin{aligned}
\frac{E_C^{(7)}}{A^{(7)}} &= -\frac{3}{128\pi^4} \left(\frac{\sqrt{ab}}{a} \right)^6 \frac{\zeta(8)}{d^7} \left[1 - \frac{5}{12} \eta^2 \frac{\zeta(6)}{\zeta(8)} + \frac{3}{64} \eta^4 \frac{\zeta(4)}{\zeta(8)} \right. \\
&\quad \left. + \frac{155}{1536\pi} \eta^6 \frac{\zeta(6)}{\zeta(8)} + \frac{35}{768\pi^2} \eta^6 \frac{\zeta(4)}{\zeta(8)} + \frac{1}{256} \eta^6 \frac{\zeta(2)}{\zeta(8)} \right], \\
\frac{E_C^{(8)}}{A^{(8)}} &= -\frac{105}{8192\pi^4} \left(\frac{\sqrt{ab}}{a} \right)^7 \frac{\zeta(9)}{d^8} \left[1 - \frac{4}{7} \eta^2 \frac{\zeta(7)}{\zeta(9)} + \frac{64}{525} \eta^4 \frac{\zeta(5)}{\zeta(9)} \right], \\
\frac{E_C^{(9)}}{A^{(9)}} &= -\frac{3}{128\pi^5} \left(\frac{\sqrt{ab}}{a} \right)^8 \frac{\zeta(10)}{d^9} \left[1 - \frac{35}{48} \eta^2 \frac{\zeta(8)}{\zeta(10)} + \frac{259}{1152} \eta^4 \frac{\zeta(6)}{\zeta(10)} \right. \\
&\quad - \frac{175}{7168} \eta^6 \frac{\zeta(4)}{\zeta(10)} - \frac{4445}{49152\pi^6} \eta^8 \frac{\zeta(8)}{\zeta(10)} - \frac{5425}{73729} \eta^8 \frac{\zeta(6)}{\zeta(10)} \\
&\quad \left. - \frac{27195}{1105920\pi^2} \eta^8 \frac{\zeta(4)}{\zeta(10)} - \frac{175}{86016} \eta^8 \frac{\zeta(2)}{\zeta(10)} \right], \\
\frac{E_C^{(10)}}{A^{(10)}} &= -\frac{945}{65536\pi^5} \left(\frac{\sqrt{ab}}{a} \right)^9 \frac{\zeta(11)}{d^{10}} \left[1 - \frac{8}{9} \eta^2 \frac{\zeta(9)}{\zeta(11)} + \frac{16}{45} \eta^4 \frac{\zeta(7)}{\zeta(11)} \right. \\
&\quad \left. - \frac{256}{3675} \eta^6 \frac{\zeta(5)}{\zeta(11)} \right], \\
\frac{E_C^{(11)}}{A^{(11)}} &= -\frac{15}{512\pi^6} \left(\frac{\sqrt{ab}}{a} \right)^{10} \frac{\zeta(12)}{d^{11}} \left[1 - \frac{21}{20} \eta^2 \frac{\zeta(10)}{\zeta(12)} + \frac{329}{640} \eta^4 \frac{\zeta(8)}{\zeta(12)} \right. \\
&\quad - \frac{3229}{23040} \eta^6 \frac{\zeta(6)}{\zeta(12)} + \frac{245}{16384} \eta^8 \frac{\zeta(4)}{\zeta(12)} \\
&\quad + \frac{10731}{131072\pi^8} \eta^{10} \frac{\zeta(10)}{\zeta(12)} + \frac{6223}{65536\pi^6} \eta^{10} \frac{\zeta(8)}{\zeta(12)} \\
&\quad + \frac{10199}{196608\pi^4} \eta^{10} \frac{\zeta(6)}{\zeta(12)} \\
&\quad \left. + \frac{22603}{1474560\pi^2} \eta^{10} \frac{\zeta(4)}{\zeta(12)} + \frac{569625}{8388608} \eta^{10} \frac{\zeta(2)}{\zeta(12)} \right].
\end{aligned} \tag{32}$$

Where $\zeta(s)$ is the Riemann zeta function and $\eta = \frac{d}{\sqrt{ab}}$. Our results are interest at the limiting case which is narrow slit is defined by $\eta = \frac{d}{\sqrt{ab}} \ll 1$ [16].

We easily analysis that in the limit $a, b \rightarrow \infty$ and $d \rightarrow 0$ ($\eta \rightarrow 0$ and $\frac{\sqrt{ab}}{a} \rightarrow 1$) which means that the surfaces between two spheres converted to the parallel plate geometry. One finds that the leading term of the Casimir energy per unit area for D dimension can be written as

$$\frac{E_C^{(D)}}{A^{(D)}} = - \frac{1}{(4\pi)^{\frac{(D+1)}{2}} d^D} \Gamma\left(\frac{(D+1)}{2}\right) \zeta(D+1). \quad (33)$$

This result is exactly the same as the scalar Casimir energy of the parallel plates for D dimension [32]. Thus our approach developed here has been the satisfactory check. As far as we know this result is obtained here for the first time.

3 CONCLUSION

In this work, we have calculated the Casimir energy between closely spaced two concentric D -dimensional spheres for a massless scalar field satisfying the Dirichlet boundary conditions. We obtain the numerical results of the Casimir energy between the closely spaced two concentric spheres in space dimension $D = 4$ up to $D = 11$. Although the sign and magnitude of the Casimir energy for the spherical shell and cavity change dramatically with the dimension, the sign of the Casimir energy between closely spaced two concentric D -dimensional spheres does not change. We observed that all spacetime dimensions give us the negative renormalized vacuum energy by quantum fluctuations between closely spaced two concentric D -dimensional spheres. This result produces to the sign of stabilazition of the Casimir energy. The condition of stability will be satisfied between $D - 1$ dimensional surfaces close to each other.

The interesting result of our calculations is that any approximation technique is not needed for our geometry. All contributions in the Casimir energy for a massless scalar field comes from the higher frequencies for fixed k between two surfaces boundary conditions. $k \rightarrow \infty$ frequency modes contribution in the Casimir energy is zero for arbitrary width of annular region between closely spaced surfaces. Moreover, we find that the Casimir energy per unit area for a massless scalar field satisfying the Dirichlet boundary conditions between closely spaced two concentric D -dimensional spheres is the same D -dimensional parallel plates in the limit case [32].

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